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Group Classification of a Class of Coupled Equations

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A complete group classification of a class of coupled equations that appear in many physical problems is presented by developing the method of preliminary group classification of Ibragimov *et al.* We give a symmetry group analysis for an interesting example.

1. INTRODUCTION

Since Lie (1881) gave a group classification for a wide class of second-order partial differential equations with two independent variables, the problem of group classification for partial differential equations has attracted the attention of both theoretical physicists and mathematicians. It is well known that the problem of group classification of a given family of equations is more complicated than the problem of calculating the symmetry groups for given equations.

Recently Ibragimov *et al.* (1991) gave a simple approach for a partial solution of group classification in terms of equivalence algebra. This method has been successfully applied to some interesting partial differential equations, for instance, a model of detonation (Ibragimov and Torrisi, 1992), the nonlinear diffusion equations (Yung *et al.*, 1994), and a binary reacting mixture (Lalicate and Torrisi, 1994).

This article extends the technique and applies it to a wide class of coupled equations that appear in dispersionless dynamic systems. The coupled equations read here

$$u_t + \frac{df}{dx} = 0 \tag{1.1a}$$

$$v_{xt} - g = 0 \tag{1.1b}$$

where f and g are arbitrary functions of u and v.

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An interesting example of (1.1) is the system

$$\Delta_1 = u_t + (v^2)_x = 0 \tag{1.2a}$$

$$\Delta_2 = v_{xt} - 2uv = 0 \tag{1.2b}$$

which describes an integrable dispersionless model, and can be solved by the inverse scattering method (Konno and Oono, 1994).

This paper is arranged as follows. In Section 2, we construct the equivalence algebra and the projective algebra L_7. The adjoint group for algebra L_7 is given in Section 3. In Section 4, we obtain the optimal system of one-dimensional subalgebras of L_7; the classification results are listed in Table II. Section 5 gives the symmetry group analysis. We end with a summary of the results.

2. EQUIVALENCE ALGEBRA AND PROJECTIVE ALGEBRA L_7

An equivalence transformation is a nondegenerate change of variables t, x, u, v in addition to a change of the functions $f(u, v)$ and $g(u, v)$.

The generator of an equivalence transformation has the form

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g} \tag{2.1}$$

where ξ^i, η^i, $i = 1, 2$, are functions of (t, x, u, v) and μ^i, $i = 1, 2$, are functions of (t, x, u, v, f, g).

Equations (1.1) can be written as

$$u_t + f_u \cdot u_x + f_v \cdot v_x = 0 \tag{2.2a}$$

$$v_{xt} - g = 0 \tag{2.2b}$$

$$f_t = f_x = 0 \tag{2.2c}$$

$$g_t = g_x = 0 \tag{2.2d}$$

The invariance conditions for (2.2) are

$$\mathrm{Pr}^{(2)}Y(u_t + f_u \cdot u_x + f_v \cdot v_x) = 0 \tag{2.3a}$$

$$\mathrm{Pr}^{(2)}Y(v_{xt} - g) = 0 \tag{2.3b}$$

$$\mathrm{Pr}^{(2)}Y(f_t) = \mathrm{Pr}^{(2)}Y(f_x) = 0 \tag{2.3c}$$

$$\mathrm{Pr}^{(2)}Y(g_t) = \mathrm{Pr}^{(2)}Y(g_x) = 0 \tag{2.3d}$$

under the conditions (2.2), where $\mathrm{Pr}^{(2)}Y$ is the second prolongation of Y.

The effective second prolongation $\mathrm{Pr}^{(2)}Y$ of Y is

$$
\begin{aligned}
\mathrm{Pr}^{(2)}Y = Y &+ \eta^{1,x}\frac{\partial}{\partial u_x} + \eta^{2,x}\frac{\partial}{\partial v_x} + \mu^{1,t}\frac{\partial}{\partial f_t} + \mu^{1,x}\frac{\partial}{\partial f_x} \\
&+ \mu^{1,v}\frac{\partial}{\partial f_v} + \mu^{1,u}\frac{\partial}{\partial f_u} + \mu^{2,t}\frac{\partial}{\partial g_t} \\
&+ \mu^{2,x}\frac{\partial}{\partial g_x} + \eta^{2,tx}\frac{\partial}{\partial v_{tx}}
\end{aligned}
\tag{2.4}
$$

The formulas for $\eta^{1,x}$, $\eta^{2,x}$, $\mu^{1,t}$, $\mu^{1,x}$, $\mu^{1,u}$, $\mu^{1,v}$, $\mu^{2,t}$, $\mu^{2,x}$, and $\eta^{2,tx}$ can be found in Olver (1986). For instance,

$$
\begin{aligned}
\eta^{2,tx} = \eta^2_{tx} &+ \eta^2_{tu}u_x + \eta^2_{tv}v_x + (\eta^2_{xu} + \eta^2_{uu}u_x + \eta^2_{uv}v_x)u_t \\
&+ \eta^2_u u_{xt} + \eta^2_{xv}v_t + \eta^2_{uv}u_x v_t + \eta^2_{vv}v_x v_t - \zeta^1_{tx}v_t \\
&+ (\eta^2_v - \zeta^1_t - \zeta^2_x)v_{tx} - \zeta^2_{tx}v_x - \zeta^2_t v_{xx} - \zeta^1_x v_{tt}
\end{aligned}
\tag{2.5}
$$

Substituting (2.4) into (2.3), and solving the overdetermined partial differential equations, we obtain

$$
\begin{aligned}
&\xi^1 = c_1 t + c_2, \qquad \xi^2 = c_3 x + c_4, \qquad \eta^1 = c_5 u + c_6, \qquad \eta^2 = c_7 v + c_8 \\
&\mu^1 = (c_3 - c_1 + c_3)f + c_9, \qquad \mu^2 = (c_7 - c_1 - c_3)g
\end{aligned}
\tag{2.6}
$$

with arbitrary constants c_i, $i = 1, 2, \ldots, 9$.

The equivalence invariant vector field can be written as

$$
\begin{aligned}
Y = (c_1 t + c_2)\frac{\partial}{\partial t} &+ (c_3 x + c_4)\frac{\partial}{\partial x} \\
&+ (c_5 u + c_6)\frac{\partial}{\partial u} + (c_7 v + c_8)\frac{\partial}{\partial v} \\
&+ [(c_5 - c_1 + c_3)f + c_9]\frac{\partial}{\partial f} + (c_7 - c_1 - c_3)g\frac{\partial}{\partial g}
\end{aligned}
\tag{2.7}
$$

Then the equivalence algebra is generated by

$$
\begin{aligned}
&Y_1 = t\frac{\partial}{\partial t} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g}, \qquad Y_2 = \frac{\partial}{\partial t}, \\
&Y_3 = x\frac{\partial}{\partial x} + f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g}, \qquad Y_4 = \frac{\partial}{\partial x} \\
&Y_5 = u\frac{\partial}{\partial u} + f\frac{\partial}{\partial f}, \qquad Y_6 = \frac{\partial}{\partial u}, \\
&Y_7 = v\frac{\partial}{\partial v} + g\frac{\partial}{\partial g}, \qquad Y_8 = \frac{\partial}{\partial v}, \qquad Y_9 = \frac{\partial}{\partial f}
\end{aligned}
\tag{2.8}
$$

Let

$$Z = \zeta^1 \frac{\partial}{\partial t} + \zeta^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} \tag{2.9}$$

$$Z = \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g} \tag{2.10}$$

To calculate the principal Lie algebra, we need the following proposition (Konno and Oono, 1994).

Proposition. An operator Z belongs to the principal Lie algebra L_σ for the system (1.1) $\leftrightarrow$ Z has one equivalence generator Y, such that

$$Z = 0 \tag{2.11}$$

In terms of above proposition, we immediately have

$$c_1 = c_3 = c_5 = c_6 = c_7 = c_8 = c_9 = 0 \tag{2.12}$$

So the principal Lie algebra $L\sigma$ is generated by

$$Z_1 = \frac{\partial}{\partial t}, \qquad Z_2 = \frac{\partial}{\partial x} \tag{2.13}$$

The functions f and g depend on u and v, and the projections of Y_i, $i = 1, \ldots, 9$, on (u, v, f, g) are

$$\begin{aligned}
Z_1 &= \mathrm{pr}(Y_1) = -f \frac{\partial}{\partial f} - g \frac{\partial}{\partial g} \\
Z_2 &= \mathrm{pr}(Y_3) = f \frac{\partial}{\partial f} - g \frac{\partial}{\partial g} \\
Z_3 &= \mathrm{pr}(Y_5) = u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f} \\
Z_4 &= \mathrm{pr}(Y_6) = \frac{\partial}{\partial u} \\
Z_5 &= \mathrm{pr}(Y_7) = v \frac{\partial}{\partial v} + g \frac{\partial}{\partial g} \\
Z_6 &= \mathrm{pr}(Y_8) = \frac{\partial}{\partial v} \\
Z_7 &= \mathrm{pr}(Y_9) = \frac{\partial}{\partial f}
\end{aligned} \tag{2.14}$$

We denote $L_7 = \{Z_i, i = 1, 2, \ldots, 7\}$.

3. ADJOINT GROUP FOR L_7

The commutation relations of $\{Z_i\}$, are summarized in Table I.

Denote by A the elements of the algebra adL_7; a basis of the algebra adL_7 is

$$A_\alpha = [Z_\alpha, Z_\beta]\frac{\partial}{\partial Z_\beta}, \qquad \alpha = 1, 2, \ldots, 7 \tag{3.1}$$

Using Table I, we obtain

$$\begin{aligned}
A_1 &= Z_1\frac{\partial}{\partial Z_7}, \qquad A_2 = -Z_7\frac{\partial}{\partial Z_7}\\
A_3 &= -Z_4\frac{\partial}{\partial Z_4} - Z_7\frac{\partial}{\partial Z_7}, \qquad A_4 = Z_4\frac{\partial}{\partial Z_3}\\
A_5 &= -Z_6\frac{\partial}{\partial Z_6}, \qquad A_6 = Z_6\frac{\partial}{\partial Z_5},\\
A_7 &= Z_7\left(\frac{\partial}{\partial Z_2} + \frac{\partial}{\partial Z_3} - \frac{\partial}{\partial Z_1}\right)
\end{aligned} \tag{3.2}$$

The element A_3 generates the one-parameter group of linear transformations

$$\begin{aligned}
&Z_1' = Z_1, \qquad Z_2' = Z_2, \qquad Z_3' = Z_3, \qquad Z_4' = (1 - a_3)Z_4\\
&Z_5' = Z_5, \qquad Z_6' = Z_6, \qquad Z_7' = (1 - a_3)Z_7
\end{aligned} \tag{3.3}$$

with arbitrary parameter a_3, which can equivalently be represented by a matrix

Table I. Commutation Relations of $\{Z_i\}$

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
Z_1	0	0	0	0	0	0	Z_7
Z_2	0	0	0	0	0	0	$-Z_7$
Z_3	0	0	0	$-Z_4$	0	0	$-Z_7$
Z_4	0	0	Z_4	0	0	$-Z_6$	0
Z_5	0	0	0	0	0	0	0
Z_6	0	0	0	0	Z_6	0	0
Z_7	$-Z_7$	Z_7	Z_7	0	0	0	0

$$M_3(a_3) = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 - a_3 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 - a_3 \end{bmatrix}, \quad -\infty < a_3 < +\infty \tag{3.4}$$

Along the same lines, we obtain $M_1(a_1)$, $M_2(a_2)$, . . . , $M_7(a_7)$, $-\infty < a_i < +\infty$, and

$$M = \prod_{i=1}^{7} M_i(a_i)$$

$$= \begin{bmatrix} 1 & 0 & & & & & -a_7 \\ & 1 & 0 & & & & a_7 \\ & & 1 & a_4 & & & a_7 \\ & & 0 & 1 - a_3 & 0 & & \\ & & & 0 & 1 & a_6 & \\ & & & & 0 & 1 - a_5 & 0 \\ & & & & & 0 & (1 + a_1)(1 - a_2)(1 - a_3) \end{bmatrix} \tag{3.5}$$

To determine the adjoint group of L_7, we require the coefficients $e = (e_i)$ of

$$Z = \sum_{i=1}^{7} e_i Z_i \tag{3.6}$$

The vector e is transformed to $\bar{e}$ by the transposed matrix M^{τ} of M, and then the transformation e has the following form:

$$\bar{e}_1 = e_1, \quad \bar{e}_2 = e_2, \quad \bar{e}_3 = e_3, \quad \bar{e}_4 = a_4 e_3 + (1 - a_3) e_4, \quad \bar{e}_5 = e_5$$

$$\bar{e}_6 = a_6 e_5 + (1 - a_5) e_6$$

$$\bar{e}_7 = (1 + a_1)(1 - a_2)(1 - a_3) e_7 + a_7 (e_2 + e_3 - e_1) \tag{3.7}$$

These transformations give rise to the adjoint group of the algebra L_7.

4. CONSTRUCTION OF THE OPTIMAL SYSTEM OF ONE-DIMENSIONAL SUBALGEBRA OF L_7

In this section, we use a general approach to construct the optimal system of the one-dimensional subalgebra of L_7. The starting point is the transformations (3.7). Notice that transformations (3.7) leave e^1, e^2, e^3, and e^5 invariant. So we have to look for all possibilities for e_1, e_2, e_3, and e_5 and in every case to simplify other components of e by means of (3.7).

First, we consider the case

$$e_1 \neq 0, \qquad e_2 \neq 0, \qquad e_3 \neq 0, \qquad e_5 \neq 0 \tag{4.1}$$

In this case, we choose

$$a_3 = a_5 = 0 \tag{4.2}$$

and

$$\bar{a}_4 = \bar{a}_6 = 0 \tag{4.3}$$

by putting

$$a_4 = -\frac{e_4}{e_3}, \qquad a_6 = -\frac{e_6}{e_5} \tag{4.4}$$

To further proceed, we distinguish the two following subcases:

$$e_2 + e_3 - e_1 \neq 0 \tag{4.5a}$$

$$e_2 + e_3 - e_1 = 0 \tag{4.5b}$$

If (4.5a) is valid, we choose

$$a_7 = \frac{e_1}{e_1 - e_2 - e_3}, \qquad a_1 = a_2 = 0 \tag{4.6}$$

Then e becomes

$$(e_1, e_2, e_3, 0, e_5, 0, 0) \tag{4.7}$$

In the subcase (4.5b), we get

$$\bar{e}_1 = (1 + a_1)(1 - a_2)e_1 \tag{4.8}$$

Therefore e is transformed to

$$(e_2 + e_3, e_2, e_3, 0, e_5, 0, (1 + a_1)(1 - a_2)e_7) \tag{4.9}$$

so we obtain the following two nonequivalent operators which correspond to (4.5a) and (4.5b):

$$Z_5 + \alpha Z_1 + \beta Z_2 + \gamma Z_3, \qquad \alpha \neq \beta + \gamma, \quad \alpha, \beta, \gamma \neq 0 \tag{4.10a}$$

$$Z_5 + (\alpha + \beta)Z_1 + \alpha Z_2 + \beta Z_3 + \gamma Z_7,$$
$$\alpha + \beta \neq 0, \quad \alpha, \beta \neq 0 \tag{4.10b}$$

The second case is

$$e_1 \neq 0, \qquad e_2 \neq 0, \qquad e_3 \neq 0, \qquad e_5 = 0 \tag{4.11}$$

Following the same procedure, we get two nonequivalent operators

$$Z_1 + \alpha Z_2 + \beta Z_3 + \gamma Z_6, \qquad \alpha \neq 0, \quad \beta \neq 0, \quad \alpha + \beta \neq 1 \tag{4.12a}$$

$$Z_1 + \alpha Z_2 + (1 - \alpha) Z_3 + \beta Z_6 + \gamma Z_7, \qquad \alpha \neq 0, 1 \tag{4.12b}$$

The other cases are

$$e_1 \neq 0, \qquad e_2 \neq 0, \qquad e_3 = 0, \qquad e_5 \neq 0 \tag{4.13}$$

$$Z_1 + \alpha Z_2 + \beta Z_4 + \gamma Z_5, \qquad \alpha \neq 0, 1, \quad \gamma \neq 0 \tag{4.14a}$$

$$Z_1 + Z_2 + \alpha Z_4 + \beta Z_5 + \gamma Z_7, \qquad \beta \neq 0 \tag{4.14b}$$

$$e_1 \neq 0, \qquad e_2 \neq 0, \qquad e_3 = e_5 = 0 \tag{4.15}$$

$$Z_1 + \alpha Z_2 + \beta Z_4 + \gamma Z_6, \qquad \alpha \neq 0, 1 \tag{4.16a}$$

$$Z_1 + Z_2 + \alpha Z_4 + \beta Z_6 + \gamma Z_7 \tag{4.16b}$$

$$e_1 \neq 0, \qquad e_2 = 0, \qquad e_3 \neq 0, \qquad e_5 \neq 0 \tag{4.17}$$

$$Z_1 + \alpha Z_3 + \beta Z_5, \qquad \alpha \neq 0, 1 \tag{4.18a}$$

$$Z_1 + Z_3 + \alpha Z_5 + \beta Z_7, \qquad \alpha \neq 0 \tag{4.18b}$$

$$e_1 \neq 0, \qquad e_2 = e_5 = 0, \qquad e_3 \neq 0 \tag{4.19}$$

$$Z_1 + \alpha Z_3 + \beta Z_6, \qquad \alpha \neq 0, 1 \tag{4.20a}$$

$$Z_1 + Z_3 + \alpha Z_6 + \beta Z_7 \tag{4.20b}$$

$$e_1 \neq 0, \qquad e_2 = e_3 = 0, \qquad e_5 \neq 0 \tag{4.21}$$

$$Z_1 + \alpha Z_4 + \beta Z_5, \qquad \beta \neq 0 \tag{4.22}$$

$$e_1 \neq 0, \qquad e_2 = e_3 = e_5 = 0 \tag{4.23}$$

$$Z_1 + \alpha Z_4 + \beta Z_6 \tag{4.24}$$

$$e_1 = 0, \qquad e_2 \neq 0, \qquad e_3 \neq 0, \qquad e_5 \neq 0 \tag{4.25}$$

$$Z_2 + \alpha Z_3 + \beta Z_5, \qquad \alpha \neq 0, -1, \quad \beta \neq 0 \tag{4.26a}$$

$$Z_2 - Z_3 + \alpha Z_5 + \beta Z_7, \qquad \alpha \neq 0 \tag{4.26b}$$

$$e_1 = e_5 = 0, \qquad e_2 \neq 0, \qquad e_3 \neq 0 \tag{4.27}$$

$$Z_2 + \alpha Z_3 + \beta Z_6, \qquad \alpha \neq 0, -1 \tag{4.28a}$$

$$Z_2 - Z_3 + \alpha Z_6 + \beta Z_7 \tag{4.28b}$$

$$e_1 = e_3 = 0, \qquad e_2 \neq 0, \qquad e_5 \neq 0 \tag{4.29}$$

$$Z_2 + \alpha Z_4 + \beta Z_5, \qquad \beta \neq 0 \tag{4.30}$$

$$e_2 \neq 0, \qquad e_1 = e_3 = e_5 = 0 \tag{4.31}$$

$$Z_2 + \alpha Z_4 + \beta Z_6 \tag{4.32}$$

$$e_1 = e_2 = 0, \qquad e_3 \neq 0, \qquad e_5 \neq 0 \tag{4.33}$$

$$Z_3 + \alpha Z_5, \qquad \alpha \neq 0 \tag{4.34}$$

$$e_1 = e_2 = e_3 = 0, \qquad e_5 \neq 0 \tag{4.35}$$

$$Z_5 + \alpha Z_4 + \beta Z_7 \tag{4.36}$$

$$e_1 = e_2 = e_5 = 0, \qquad e_3 \neq 0 \tag{4.37}$$

$$Z_6 \tag{4.38}$$

$$e_1 = e_2 = e_3 = e_5 = 0 \tag{4.39}$$

$$\begin{aligned} &Z_4, Z_7, Z_4 + Z_6, Z_4 - Z_6, Z_4 + Z_7, Z_4 - Z_7, Z_6 + Z_7, \\ &Z_6 - Z_7, Z_4 + Z_6 + Z_7, Z_4 + Z_6 - Z_7, \\ &Z_4 - Z_6 + Z_7, Z_4 - Z_6 - Z_7 \end{aligned} \tag{4.40}$$

Summarizing the above results, we obtain the following optimal system of one-dimensional subalgebrals of L_7:

$$Z^{(1)} = Z_5 + \alpha Z_1 + \beta Z_2 + \gamma Z_3 \qquad (\alpha, \beta, \gamma \neq 0, \alpha \neq \beta + \gamma)$$

$$Z^{(2)} = Z_5 + (\alpha + \beta)Z_1 + \alpha Z_2 + \beta Z_3 + \gamma Z_7 \qquad (\alpha \neq 0, \beta \neq 0)$$

$$Z^{(3)} = Z_1 + \alpha Z_2 + \beta Z_3 + \gamma Z_6 \qquad (\alpha, \beta \neq 0, \alpha + \beta \neq 1)$$

$$Z^{(4)} = Z_1 + \alpha Z_2 + (1 - \alpha)Z_3 + \beta Z_6 + \gamma Z_7 \qquad (\alpha \neq 0, 1)$$

$$Z^{(5)} = Z_1 + \alpha Z_2 + \beta Z_4 + \gamma Z_5 \qquad (\alpha \neq 0, 1, \gamma \neq 0)$$

$$Z^{(6)} = Z_1 + Z_2 + \alpha Z_4 + \beta Z_5 + \gamma Z_7 \qquad (\beta \neq 0)$$

$$Z^{(7)} = Z_1 + \alpha Z_2 + \beta Z_4 + \gamma Z_6 \qquad (\beta \neq 0, \alpha \neq 0, 1)$$

$$Z^{(8)} = Z_1 + \alpha Z_2 + \gamma Z_6 \qquad (\gamma \neq 0)$$

$$Z^{(9)} = Z_1 + Z_2 + \alpha Z_4 + \beta Z_6 + \gamma Z_7 \qquad (\alpha \neq 0)$$

$$Z^{(10)} = Z_1 + Z_2 + \beta Z_6 + \gamma Z_7 \qquad (\beta \neq 0)$$

$$Z^{(11)} = Z_1 + \alpha Z_3 + \beta Z_5 \qquad (\alpha \neq 0, 1, \beta \neq 0)$$

$$Z^{(12)} = Z_1 + Z_3 + \alpha Z_5 + \beta Z_7 \qquad (\alpha \neq 0)$$

$$Z^{(13)} = Z_1 + \alpha Z_3 + \beta Z_6 \qquad (\alpha \neq 0,\ 1)$$
$$Z^{(14)} = Z_1 + Z_3 + \alpha Z_6 + \beta Z_7$$
$$Z^{(15)} = Z_1 + \alpha Z_4 + \beta Z_5 \qquad (\beta \neq 0)$$
$$Z^{(16)} = Z_1 + \alpha Z_4 + \beta Z_6 \qquad (\beta \neq 0)$$
$$Z^{(17)} = Z_2 + \alpha Z_3 + \beta Z_5 \qquad (\alpha \neq 0,\ -1,\ \beta \neq 0)$$
$$Z^{(18)} = Z_2 - Z_3 + \alpha Z_5 + \beta Z_7$$
$$Z^{(19)} = Z_2 + \alpha Z_3 + \beta Z_6 \qquad (\alpha \neq 0,\ -1)$$
$$Z^{(20)} = Z_2 - Z_3 + \alpha Z_6 + \beta Z_7$$
$$Z^{(21)} = Z_2 + \alpha Z_4 + \beta Z_5$$
$$Z^{(22)} = Z_2 + \alpha Z_4 + \beta Z_6 \qquad (\alpha \neq 0)$$
$$Z^{(23)} = Z_2 + \beta Z_6 \qquad (\beta \neq 0)$$
$$Z^{(24)} = Z_3 + \alpha Z_5$$
$$Z^{(25)} = Z_5 + \alpha Z_4 + \beta Z_7$$
$$Z^{(26)} = Z_6$$
$$Z^{(27)} = Z_4$$
$$Z^{(28)} = Z_7$$
$$Z^{(29)} = Z_4 + Z_6$$
$$Z^{(30)} = Z_4 - Z_6$$
$$Z^{(31)} = Z_4 + Z_7$$
$$Z^{(32)} = Z_4 - Z_7$$
$$Z^{(33)} = Z_6 + Z_7$$
$$Z^{(34)} = Z_6 - Z_7$$
$$Z^{(35)} = Z_4 + Z_6 + Z_7$$
$$Z^{(36)} = Z_4 + Z_6 - Z_7$$
$$Z^{(37)} = Z_4 - Z_6 + Z_7$$
$$Z^{(38)} = Z_4 - Z_6 - Z_7 \tag{4.41}$$

Applying the above results, we can give a complete classification admitting an extension by one of the principal Lie algebra for (1.1). The results are listed in Table II.

5. SYMMETRY GROUP APPROACH FOR (1.2)

The symmetry algebra of (1.2) consists of differential operators of the form

$$X = \xi(t, x, u, v) \frac{\partial}{\partial t} + \eta(t, x, u, v) \frac{\partial}{\partial x} + \phi(t, x, u, v) \frac{\partial}{\partial u} + \Psi(t, x, u, v) \frac{\partial}{\partial v} \tag{5.1}$$

such that their second prolongation satisfies

$$\mathrm{Pr}^{(2)} X(\Delta_i)\big|_{\Delta_i=0, i=1,2} = 0, \qquad i = 1, 2 \tag{5.2}$$

This condition is imposed by application of the differential operator $\mathrm{Pr}^{(2)}$ to Δ_i and then elimination. Equating to zero the coefficients of linearly independent expressions in the t and x derivatives of u and v, we obtain a system of determining equations for the coefficients ζ, η, ϕ, and Ψ in (5.1). The general element of the symmetry algebra of (1.2) has the form

$$X = \frac{\partial}{\partial t}, \qquad Y = t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}$$

$$Z(\sigma) = \sigma(x) \frac{\partial}{\partial x} - \sigma_x(x) u \frac{\partial}{\partial u} \tag{5.3}$$

where $\sigma(x)$ is an arbitrary function of x. We thus see that this Lie algebra is indeed infinite dimensional, as its element is labeled by an arbitrary function $\sigma(x)$.

The commutation relations of (5.3) are

$$[X, Y] = X, \qquad [Y, Z] = [X, Z] = 0$$

$$[Z(\sigma_1), Z(\sigma_2)] = Z(\sigma_1 \dot{\sigma}_2 - \sigma_2 \dot{\sigma}_1) \tag{5.4}$$

which shows that $\{X, Y, Z\}$ has a Kac–Moody–Virasoro structure.

We now look for the particular solutions of (1.2) that are invariant under the subgroups of the symmetry groups which correspond to the Lie algebras (5.3). To be clear, we distinguish the following cases.

Case (i). The subalgebra $Y + Z(x)$.

The solutions of (1.2) have the form

$$u = x^{-2} f(I), \qquad v = x^{-1} g(I), \qquad I = t/x \tag{5.5}$$

Table II. The Classification $Q_i(I)$, $i = 1, 2$, as a Function of I

No.	Z	Invariant I	Equation	Additional operator	Remark
1	$Z^{(1)}$	v^r/u	$f = v^{r+\beta-\alpha}Q_1(I)$, $g = v^{-(\alpha+\beta)}Q_2(I)$	$\alpha t\frac{\partial}{\partial t} + \beta x\frac{\partial}{\partial x} + \gamma u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$	$\alpha, \beta, \gamma \neq 0$, $\alpha \neq \beta + \gamma$
2	$Z^{(2)}$	v^β/u	$f = \gamma \ln v + Q_1(I)$, $g = v^{1-2\alpha-\beta}Q_1(I)$	$(\alpha + \beta)t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} + \beta u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$	$\alpha \neq 0, \beta \neq 0$
3	$Z^{(3)}$	$u^{-\gamma/\beta}e^v$	$f = u^{-(\alpha+1)/\beta}Q_1(I)$, $g = u^{(\alpha+\beta-1)/\beta}Q_2(I)$	$t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} + \beta u\frac{\partial}{\partial u} + \gamma\frac{\partial}{\partial v}$	$\alpha, \beta \neq 0$, $\alpha + \beta \neq 1$
4	$Z^{(4)}$	$u^{\beta/(\alpha-1)}e^v$	$f = \frac{\gamma}{1-\alpha}\ln u + Q_1(I)$, $g = u^{(\alpha+1)/(\alpha-1)}Q_2(I)$	$t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} + (1-\alpha)\frac{\partial}{\partial u} + \beta v\frac{\partial}{\partial v}$	$\alpha \neq 0, 1$
5	$Z^{(5)}$	$ve^{-(\gamma/\beta)u}$	$f = v^{(\alpha-1)/\gamma}Q_1(I)$, $g = v^{(r-\alpha-1)/\gamma}Q_2(I)$	$t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial u} + \gamma v\frac{\partial}{\partial v}$	$\alpha \neq 0, 1$, $\gamma \neq 0$
6	$Z^{(6)}$	$ve^{-(\beta/\alpha)u}$	$f = \frac{\gamma}{\alpha}u + Q_1(I)$, $g = v^{(\beta-2)/\beta}Q_2(I)$	$t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial u} + \beta v\frac{\partial}{\partial v}$	$\beta \neq 0, \alpha \neq 0$
7	$Z^{(7)}$	$\gamma u - \beta v$	$f = e^{[(\alpha-1)/\beta]u}Q_1(I)$, $g = e^{-[(\alpha+1)/\beta]u}Q_2(I)$	$t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial u} + \gamma\frac{\partial}{\partial v}$	$\beta \neq 0$, $\alpha \neq 0, 1$
8	$Z^{(8)}$	γu	$f = e^{[(\alpha-1)/\gamma]u}Q_1(I)$, $g = e^{-[(\alpha+1)/\gamma]u}Q_2(I)$	$t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial v}$	$\gamma \neq 0$
9	$Z^{(9)}$	$\beta u - \alpha v$	$f = \frac{\gamma u}{\alpha} + Q_1(I)$, $g = e^{-(2/\alpha)u}Q_2(I)$	$t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial v}$	$\alpha \neq 0$
10	$Z^{(10)}$	βu	$f = \frac{\gamma v}{\beta} + Q_1(I)$, $g = e^{-(2/\beta)v}Q_2(I)$	$t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial v}$	$\beta \neq 0$
11	$Z^{(11)}$	$v^\alpha u^{-\beta}$	$f = u^{(\alpha-1)/\alpha}Q_1(I)$, $g = v^{(\beta-1)/\beta}Q_2(I)$	$t\frac{\partial}{\partial t} + \alpha u\frac{\partial}{\partial u} + \beta v\frac{\partial}{\partial v}$	$\alpha \neq 0, 1$, $\beta \neq 0$
12	$Z^{(12)}$	u^α/v	$f = \beta \ln u + Q_1(I)$, $g = u^{\alpha-1}Q_2(I)$	$t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + \alpha v\frac{\partial}{\partial v}$	$\alpha \neq 0$

Table II. Continued

No.	Z	Invariant I	Equation	Additional operator	Remark
13	$Z^{(13)}$	$u^{-\beta}e^{\alpha v}$	$f = u^{(\alpha-1)/\alpha}Q_1(I)$ $g = u^{-1/\alpha}Q_2(I)$	$t\frac{\partial}{\partial t} + \alpha u\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial v}$	$\alpha \neq 0, 1$
14	$Z^{(14)}$	$u^{\alpha}e^{-v}$	$f = \beta \ln u + Q_1(I)$ $g = u^{-1}Q_2(I)$	$t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + \alpha\frac{\partial}{\partial v}$	
15	$Z^{(15)}$	$v^{-\alpha/\beta}e^{u}$	$f = v^{-1/\beta}Q_1(I)$ $g = v^{(\beta-1)/\beta}Q_2(I)$	$t\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial u} + \beta v\frac{\partial}{\partial v}$	$\beta \neq 0$
16	$Z^{(16)}$	$\alpha v - \beta u$	$f = e^{-v/\beta}Q_1(I)$ $g = e^{-v/\beta}Q_2(I)$	$t\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial v}$	$\beta \neq 0$
17	$Z^{(17)}$	$u^{\beta}e^{-\alpha v}$	$f = u^{(\alpha+1)/\alpha}Q_1(I)$ $g = u^{(\beta-1)/\alpha}Q_2(I)$	$x\frac{\partial}{\partial x} + \alpha u\frac{\partial}{\partial u} + \beta v\frac{\partial}{\partial v}$	$\alpha \neq 0, -1,$ $\beta \neq 0$
18	$Z^{(18)}$	vu^{α}	$f = -\ln u + Q_1(I)$ $g = u^{1-\alpha}Q_2(I)$	$x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} + \alpha v\frac{\partial}{\partial v}$	
19	$Z^{(19)}$	$e^{v}u^{-\beta/\alpha}$	$f = u^{(\alpha+1)/\alpha}Q_1(I)$ $g = u^{-1/\alpha}Q_2(I)$	$x\frac{\partial}{\partial x} + \alpha u\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial v}$	$\alpha \neq 0, -1$
20	$Z^{(20)}$	$e^{v}u^{\alpha}$	$f = -\beta \ln u + Q_1(I)$ $g = uQ_2(I)$	$x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} + \alpha\frac{\partial}{\partial v}$	
21	$Z^{(21)}$	$e^{u}v^{-\alpha/\beta}$	$f = v^{1/\beta}Q_1(I)$ $g = v^{(\beta-1)/\beta}Q_2(I)$	$x\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial u} + \beta v\frac{\partial}{\partial v}$	$\beta \neq 0$
22	$Z^{(22)}$	$\alpha v - \beta u$	$f = e^{u/\alpha}Q_1(I)$ $g = e^{-u/\alpha}Q_2(I)$	$x\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial v}$	$\alpha \neq 0$
23	$Z^{(23)}$	$-\beta u$	$f = e^{v/\beta}Q_1(I)$ $g = e^{-v/\beta}Q_2(I)$	$x\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial v}$	$\beta \neq 0$
24	$Z^{(24)}$	$vu^{-\alpha}$	$f = uQ_1(I)$ $g = u^{\alpha}Q_2(I)$	$u\frac{\partial}{\partial u} + \alpha v\frac{\partial}{\partial v}$	$\alpha \neq 0$
25	$Z^{(25)}$	$e^{u}v^{-\alpha}$	$f = \beta \ln v + Q_1(I)$ $g = vQ_2(I)$	$v\frac{\partial}{\partial v} + \alpha\frac{\partial}{\partial u}$	
26	$Z^{(26)}$	u	$f = Q_1(I)$ $g = Q_2(I)$	$\frac{\partial}{\partial v}$	

Table II. Continued

No.	Z	Invariant I	Equation	Additional operator	Remark
27	$Z^{(27)}$	v	$f = Q_1(I)$ $g = Q_2(I)$	$\frac{\partial}{\partial u}$	
28	$Z^{(29)}$	$u - v$	$f = Q_1(I)$ $g = Q_2(I)$	$\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$	
29	$Z^{(30)}$	$u + v$	$f = Q_1(I)$ $g = Q_2(I)$	$\frac{\partial}{\partial u} - \frac{\partial}{\partial v}$	
30	$Z^{(31)}$	v	$f = u + Q_1(I)$ $g = Q_2(I)$	$\frac{\partial}{\partial u}$	
31	$Z^{(32)}$	v	$f = -u + Q_1(I)$ $g = Q_2(I)$	$\frac{\partial}{\partial u}$	
32	$Z^{(33)}$	u	$f = v + Q_1(I)$ $g = Q_2(I)$	$\frac{\partial}{\partial v}$	
33	$Z^{(34)}$	u	$f = -v + Q_1(I)$ $g = Q_2(I)$	$\frac{\partial}{\partial v}$	
34	$Z^{(35)}$	$u - v$	$f = u + Q_1(I)$ $g = Q_2(I)$	$\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$	
35	$Z^{(36)}$	$u - v$	$f = -u + Q_1(I)$ $g = Q_2(I)$	$\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$	
36	$Z^{(37)}$	$u + v$	$f = u + Q_1(I)$ $g = Q_2(I)$	$\frac{\partial}{\partial u} - \frac{\partial}{\partial v}$	
37	$Z^{(38)}$	$u + v$	$f = -u + Q_1(I)$ $g = Q_2(I)$	$\frac{\partial}{\partial u} - \frac{\partial}{\partial v}$	

Substituting (5.5) into (1.2), we find that $f(I)$ and $g(I)$ satisfy

$$Ig'' + 2g' + 2fg = 0 \tag{5.6a}$$

$$f' - 2g^2 - 2Igg' = 0 \tag{5.6b}$$

Eliminating f, g satisfies

$$Igg''' + 3gg'' - 2g'^2 - Ig'g'' + 4g^4 + 4Ig^3g' = 0 \tag{5.7}$$

Using the transformation

$$g = \frac{1}{2}\tilde{g} \tag{5.8}$$

we find that (5.7) becomes

$$I\tilde{g}\tilde{g}''' + 3\tilde{g}\tilde{g}'' - 2\tilde{g}'^2 - I\tilde{g}'\tilde{g}'' + \tilde{g}^4 + I\tilde{g}^3\tilde{g}' = 0 \tag{5.9}$$

This equation can be reduced from dispersive long-wave equations in two spatial dimensions.

Case (*ii*). The subalgebra $Y - Z(1)$.

In this case, the solutions of (1.2) have the form

$$u = f(I), \qquad v = xg(I), \qquad I = xt \tag{5.10}$$

f and g satisfy

$$f' + 2g^2 + 2Igg' = 0 \tag{5.11a}$$

$$Ig'' + 2g' - 2fg = 0 \tag{5.11b}$$

It's easily seen that f can be determined by g, and g also satisfies (5.7).

Case (*iii*). The subalgebra of $aZ - z(1)$.

The solution in this case is a so-called one-soliton solution, which takes the form

$$u = \frac{a}{2}[1 - 2\,\mathrm{sech}^2(t + ax + b)] \tag{5.12a}$$

$$v = -\mathrm{sech}(t + ax + b) \tag{5.12b}$$

where a, b are two arbitrary constants.

6. SUMMARY AND DISCUSSIONS

We have shown the complete group classification of the coupled PDEs (1.1) admitting an extension by one of the principal Lie algebra. Moreover, we have constructed the symmetry algebra for coupled integrable dispersionless equations (1.2) and obtained the similarity reductions, which yield an interesting ordinary differential equation, which also can be reduced from a dispersive long-wave equation in two spatial dimensions.

It is also worth mentioning that since there are an infinity of conserved quantities for (1.2), a natural and interesting problem is how to get more symmetries for (1.2). Second, since (1.2) is a integrable model, do there exist other classes of partial differential equations in Table II which are integrable? We leave these problems for future study.

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